CSCI 7000 Fall 2023: Problems on Exponential Generating Functions

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Recall: the *exponential generating function* of a sequence a_n is the power series $\sum_{n \geq 0} \frac{a_n}{n!} x^n$.

1. Let $a_n =$ the number of permutations of a set of size n. What is the exponential generating function for a_n? (*Hint:* it has a closed form as a rational function.)

2. A *(labeled) cycle* of length n can be thought of in at least two ways: Way 1, as an assignment of the numbers $\{1, \ldots, n\}$, each exactly once, to the vertices of the cycle graph on n vertices; Way 2: as an equivalence class of labeled sequences, where two such sequences are considered equivalent if one can be gotten from the other by cyclically shifting the positions of the numbers (when a number "shifts" of the end of the list, it reappears at the beginning). We denote cycles with parenthesis, e.g. $(1, 2, 3)$ and $(3, 1, 2)$ are *the same cycle*, but $(1, 3, 2)$ is different.

   Let $c_n =$ the number of cycles of length n. ($c_0 = 0, c_1 = 1$.) What is the exponential generating function for c_n? What analytic function is this the power series of?

3. Suppose $\mathcal{S}_n$ is a set of labeled objects of "size" n, for each $n = 0, 1, 2, 3, \ldots$, and suppose $A(x)$ is the exponential generating function for the sequence $|\mathcal{S}_n|$. Using the product rule for exponential generating functions, prove that the exponential generating function for labeled *sets* of elements of $\mathcal{S} = \bigcup_{n \geq 0} \mathcal{S}_n$ is $\exp(A(x))$.

4. A *labeled sequence* of length n is a list of length n, containing each of the integers $\{1, \ldots, n\}$ exactly once. We denote labeled sequences

using square brackets, e.g. $[1,2,3]$ and $[3,1,2]$ are two distinct labeled sequences.

A *labeled set of cycles* of total length n is a set of cycles (of varying lengths), the sum of whose lengths is n, such that each cycle is labeled by a disjoint set of numbers, and the union of the labels is $\{1,\ldots,n\}$. We denote sets of cycles by concatenating cycles, e.g. $(1,2,3)(4,5)$ denotes a set of cycles of total length 5, consisting of two cycles, one a 3-cycle, and one a 2-cycle. Note that the ordering of cycles doesn't matter, that is, $(1,2,3)(4,5) = (4,5)(1,2,3) = (4,5)(3,1,2)$.

Give a bijection between labeled sequences of length n, and labeled sets of cycles of total length n.

5. Use the results of Questions 4, 3, and 1 to give an alternative, very quick, generating-function derivation of the answer to Question 2.

Resources

- Generatingfunctionology pp. 40–42 for rules for composing exponential generating functions.
- Flajolet & Sedgewick Chapter II for labeled structures and operations on exponential generating functions. Especially pp. 102–103 for sequences, sets, and cycles, and Section II.4 for permutations.